Approximating {0, 1, 2}-Survivable Networks with Minimum Number of Steiner Points

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Abstract. We consider low connectivity variants of the Survivable Network with Minimum Number of Steiner Points (SN-MSP) problem: given a finite set R of terminals in a metric space (M, d), a subset $B \subseteq R$ of "unstable" terminals, and connectivity requirements $\{r_{uv} : u, v \in R\}$, find a minimum size set $S \subseteq M$ of additional points such that the unit-disc graph of $R \cup S$ contains r_{uv} pairwise internally edge-disjoint and $(B \cup S)$-disjoint uv-paths for all $u, v \in R$. The case when $r_{uv} = 1$ for all $u, v \in R$ is the Steiner Tree with Minimum Number of Steiner Points (ST-MSP) problem, and the case $r_{uv} \in \{0, 1\}$ is the Steiner Forest with Minimum Number of Steiner Points (SF-MSP) problem. Let Δ be the maximum number of points in a unit ball such that the distance between any two of them is larger than 1. It is known that $\Delta = 5$ in $\mathbb{R}^2$. The previous known approximation ratio for ST-MSP was $\lfloor (\Delta + 1)/2 \rfloor + 1 + \epsilon$ in an arbitrary normed space [19], and $2.5 + \epsilon$ in the Euclidean space $\mathbb{R}^2$ [5]. Our approximation ratio for ST-MSP is $1 + \ln(\Delta - 1) + \epsilon$ in an arbitrary normed space, which in $\mathbb{R}^2$ reduces to $1 + \ln 4 + \epsilon < 2.3863 + \epsilon$. For SN-MSP with $r_{uv} \in \{0, 1, 2\}$, we give a simple Δ-approximation algorithm. In particular, for SF-MSP, this improves the previous ratio 2Δ.

1 Introduction

1.1 Problems considered

A large research effort is focused on developing algorithms for finding a "cheap" network that satisfies a certain property. In wired networks, where connecting any two nodes incurs a cost, many problems can be cast as finding a subgraph of minimum cost that satisfies some prescribed connectivity requirements. Following previous work on min-cost connectivity problems, we use the following generic notion of connectivity.

Definition 1. *Let $G = (V, E)$ be a graph and let $Q \subseteq V$. The Q-connectivity $\lambda_G^Q(u, v)$ of u, v in G is the maximum number of pairwise $(E \cup Q \setminus \{u, v\})$-disjoint uv-paths in G. Given connectivity requirements $r = \{r_{uv} : u, v \in R \subseteq V\}$ on a subset $R \subseteq V$ of terminals, we denote by $D_r = \{uv : u, v \in R, r_{uv} > 0\}$ the set of "demand edges" of r. We say that G is (r, Q)-connected, or simply r-connected if Q is understood, if $\lambda_G^Q(u, v) \geq r_{uv}$ for all $uv \in D_r$.*

Note that edge-connectivity is the case $Q = \emptyset$ and node-connectivity is the case $Q = V$. The members of $E \cup Q$ will be called *elements*, hence $\lambda_G^Q(u, v)$ is the maximum number of pairwise internally element-disjoint uv-paths in G. Variants of the following classic problem were extensively studied in the literature.

Survivable Network (SN)
Instance: A graph $G = (V, E)$ with edge costs, $Q \subseteq V$, and connectivity requirements $r = \{r_{uv} : uv \in R \subseteq V\}$.
Objective: Find a minimum-cost (r, Q)-connected subgraph H of G.

In practical networks the connectivity requirements are rather small, usually $r_{uv} \in \{0, 1, 2\}$ – so called $\{0, 1, 2\}$-SN. Particular cases in this setting are Minimum Spanning Tree (MST) ($r_{uv} = 1$ for all $u, v \in V$), Steiner Tree ($r_{uv} = 1$ for all $u, v \in R$) and Steiner Forest ($r_{uv} \in \{0, 1\}$ for all $u, v \in R$), and 2-Connected Subgraph ($r_{uv} = 2$ for all $u, v \in V$).

In wireless networks, the range and the location of the transmitters determines the resulting communication network. We consider adding a minimum number of transmitters such that the resulting communication network is (r, Q)-connected. If the range of the transmitters is fixed, our goal is to add a minimum number of transmitters, and we get the following type of problems.

Definition 2. *Let (M, d) be a metric space and let $V \subseteq M$. The* unit-disk graph *of V has node set V and edge set $\{uv : u, v \in V, d(u, v) \leq 1\}$.*

Survivable Network with Minimum Number of Steiner Points (SN-MSP)
Instance: A finite set $R \subseteq M$ of *terminals* in a metric space (M, d), a set $B \subseteq R$ of "unstable" terminals, connectivity requirements $\{r_{uv} : uv \in R\}$.
Objective: Find a minimum size set $S \subseteq M$ such that the unit-disk graph of $R \cup S$ is (r, Q)-connected, where $Q = B \cup S$.

As in previous work, we will allow to place several points at the same location, and assume that the maximum distance between terminals is polynomial in the number of terminals.

1.2 Previous work and our results

On previous work on high connectivity variants of SN problem we refer the reader to a survey in [17] and here only mention some work relevant to this paper. The Steiner Tree problem was studied extensively, c.f. [24,25,23,20,2,9], and the currently best approximation ratio for it is $\ln 4 + \epsilon$ [2]. Let τ^* denote the optimum value of a standard cut-LP relaxation for SN (see Section 3). In [10] is given a combinatorial primal-dual algorithm for Steiner Forest that computes a solution of cost at most $2\tau^*$. For $\{0, 1, 2\}$-SN a similar results is achieved by the iterative rounding method [8]; a combinatorial primal-dual algorithm that computes a solution of cost at most $3\tau^*$ is given in [21].

We survey some relevant literature on SN-MSP problems. ST-MSP is NP-hard even in $\mathbb{R}^2$, and arises in various wireless network design problems, c.f.

[1,3,4,5,12,13,18,19] for only a sample of papers in the area, where it is studied both in $\mathbb{R}^2$ and in general metric spaces. In the latter case, the approximation ratio is usually expressed in terms of the following parameter. Let Δ be the maximum number of "independent" points in the unit ball, such that the distance between any two of them is larger than 1. It is known [22] that Δ equals the maximum degree of a minimum-degree Minimum Spanning Tree in the normed space. For Euclidean distances we have $\Delta = 5$ in $\mathbb{R}^2$ and $\Delta = 11$ in $\mathbb{R}^3$, and in $\mathbb{R}^\ell$ Δ is at most the Hadwiger number [22]; hence $\Delta \leq 2^{0.401\ell(1+o(1))}$, by [11].

In finite metric spaces, ST-MSP is equivalent to the variant of the Node Weighted Steiner Tree problem when all terminals have costs 0 and the other nodes have cost 1. Klein and Ravi [16] proved that this variant is Set-Cover hard to approximate, and gave an $O(\ln|R|)$-approximation algorithm for general weights. Hence up to constants, even for finite metric spaces, the ratio $O(\ln|R|)$ of [16] is the best possible unless P=NP. Note however, that this does not exclude constant ratios for metric spaces with small Δ, e.g., $\Delta = 5$ in $\mathbb{R}^2$.

Most algorithms for SN-MSP problems applied the following reduction method, by solving the corresponding SN instance obtained as follows.

Definition 3. *Given a finite set R of points in a metric space (M, d) and an integer $k \geq 1$, the (multi)graph K_R has node set R and k parallel edges between every pair of nodes. The costs of the k edges between u, v are defined as follows. Let $\hat{d}_{uv} = \max\{\lceil d(u,v) \rceil - 1, 0\}$. If $\hat{d}_{uv} > 0$, then all the k edges have cost $\hat{d}_{uv}$. If $\hat{d}_{uv} = 0$, then one edge has cost 0 and the others have cost 1.*

Let opt denote the optimal solution value of a problem instance at hand. It is easy to see that any solution of cost C to the corresponding SN instance with $k = \max_{uv \in D_r} r_{uv}$ defines a solution S of size C to the original SN-MSP instance, where every node in S has degree exactly 2; such a solution is called a *bead solution*. Conversely, any bead solution S can be converted into a solution to the SN instance of cost at most $|S|$ (see [12,3]). Due to this bijective correspondence, we simply define a bead solution as a solution to the corresponding SN instance, and denote the optimal value of a bead solution to an instance I by $\tau = \tau(I)$. If the SN instance admits a ρ-approximation algorithm, and if for the given SN-MSP instance there exists a bead solution S of size $\leq \alpha$opt, then we get a $\rho\alpha$-approximation algorithm for the SN-MSP instance. Equivalently, for a class $\mathcal{I}$ of SN-MSP instances, define a parameter α by $\alpha = \alpha(\mathcal{I}) = \sup_{I \in \mathcal{I}} \frac{\mathsf{opt}(I)}{\tau(I)}$. Then approximation ratio ρ for SN instances that correspond to the class $\mathcal{I}$ implies approximation ratio $\alpha\rho$ for SN-MSP instances in class $\mathcal{I}$.

Măndoiu and Zelikovsky [18] showed that for ST-MSP $\alpha = \Delta - 1$. Since the instance of SN that corresponds to ST-MSP is the MST problem that can be solved in polynomial time, this gives a $(\Delta - 1)$-approximation algorithm for ST-MSP. A more general method, uses a reduction to the Minimum k-Connected Spanning Subhypergraph problem, see Section 2. This method was initiated by Zelikovsky [24], improved in a long series of papers (part of them are [24,20,23]), and culminated in the paper of Byrka, Grandoni, Rothvoß, and Sanità [2]. For ST-MSP in $\mathbb{R}^2$, Chen and Du [5] applied this method to get the currently best

known ratio $2.5+\epsilon$. In arbitrary metric spaces, the ratio $\Delta-1$ of [18] was improved to $\lfloor(\Delta+1)/2\rfloor+1+\epsilon$ in [19], also using the same method. These works assume that ST-MSP instances with a constant number of terminals can be solved in polynomial time, which holds in $\mathbb{R}^2$ if the maximum distance between terminals is polynomial in the number of terminals, see [4, Lemma 11] and the discussion there. In this paper we apply a variant due to Zelikovsky [25], and obtain the following result.

Theorem 1. ST-MSP *with constant Δ admits an approximation scheme with ratio $1+\ln(\Delta-1)+\epsilon$, provided that* ST-MSP *instances with a constant number of terminals can be solved in polynomial time. In particular, in $\mathbb{R}^2$ the ratio is $1+\ln 4+\epsilon < 2.3863+\epsilon$.*

We now discuss SN-MSP problems with $k = \max_{uv\in V} r_{uv} \geq 2$. Bredin, Demaine, Hajiaghayi, and Rus [1] considered a related problem of adding a minimum size S such that the unit disc graph of $R\cup S$ is k-node-connected (note that we require k-connectivity only between terminals). For this problem in $\mathbb{R}^2$, they gave an $O(k^5)$-approximation algorithm, but essentially they implicitly proved that for this class of problems $\alpha = O(\Delta k^3)$. Recently, it was shown in [19] that $\alpha = \Theta(\Delta k^2)$ for node-connectivity SN-MSP instances in any normed space.

Kashyap, Khuller, and Shayman [13] considered the 2-edge/node-connectivity version of SN-MSP, where $r_{uv} = 2$ for all $u,v \in R$. They used the reduction method described in Definition 3, namely, their algorithm constructs an SN instance as in Definition 3 and then converts its solution into a bead solution to the SN-MSP instance. Although they analyzed a performance of specific 2-approximation algorithms – the algorithm of Khuller and Vishkin [15] for 2-edge-connectivity and the algorithm of Khuller and Raghavachari [14] for 2-node-connectivity, they essentially proved that $\alpha = \Delta$ in both cases. This implies ratio 2Δ in both cases. The analysis of these specific algorithms was recently improved by Calinescu [3], showing that their tight performance is Δ for node-connectivity and $2\Delta-1$ for edge-connectivity. Note that the edge-connectivity version is *not* included in our model, since in our SN-MSP instances every non-terminal node is in Q, namely, the paths are required to be S disjoint.

Let $\tau^* = \tau^*(I)$ denote the optimal value of a *fractional* bead solution of an SN-MSP instance I, namely, τ^* is the optimum of a standard cut-LP relaxation for the corresponding SN instance (see Section 3). Here we observe, that if the algorithm we use for the corresponding SN instance computes a solution of cost at most $\rho\tau^*$, then the relevant parameter is the following.

Definition 4. *For a class $\mathcal{I}$ of* SN-MSP *instances, let $\alpha^* = \alpha^*(\mathcal{I}) = \sup_{I\in\mathcal{I}} \frac{\mathrm{opt}(I)}{\tau^*(I)}$.*

Theorem 2. *For Q-connectivity $\{0,1,2\}$-SN-MSP $\alpha^* = \frac{\Delta}{2}$. Thus if Q-connectivity $\{0,1,2\}$-SN admits a polynomial time algorithm that computes a solution of cost at most $\rho\tau^*$, then Q-connectivity $\{0,1,2\}$-SN-MSP admits approximation ratio $\rho \cdot \frac{\Delta}{2}$. In particular, for $\rho = 2$ the ratio is Δ, and thus $\{0,1,2\}$-SN-MSP admits a Δ-approximation algorithm.*

Theorems 1 and 2 are proved in Sections 2 and 3, respectively.

2 Proof of Theorem 1

We consider a generic problem defined in [19], that includes both ST-MSP and the classic Steiner Tree problem.

Generalized Steiner Tree

Instance: A (possibly infinite) graph $G = (V, E)$, a finite set $R \subseteq V$ of terminals, and a monotone subadditive cost function c on subgraphs of G.

Objective: Find a minimum-cost connected finite subtree T of G containing R.

Instead of considering optimal connections only between pairs of terminals, we consider optimal connections of terminal subsets of size at most k.

Definition 5. *For an instance of* Generalized Steiner Tree *and an integer k, $2 \le k \le |R|$, the hypergraph $\mathcal{H}_k = (R, \mathcal{E}_k)$ has node set R and hyperedge set $\mathcal{E}_k = \{A \subseteq R : 2 \le |A| \le k\}$. The cost $c^*(A)$ of $A \in \mathcal{E}_k$ is the cost of an optimal solution T_A to the* Generalized Steiner Tree *instance with terminal set A.*

Given a hypergraph $\mathcal{H}$ with hyperedge costs, the Minimum Connected Spanning Sub-hypergraph problem seeks a minimum cost subset of hyperedges that connects any two nodes. The construction in Definition 5 converts the Generalized Steiner Tree problem into the Minimum Connected Spanning Sub-hypergraph problem in a hypergraph $\mathcal{H}_k$ of rank k. Any solution of cost C to this problem correspond to a solution of value at most C to Generalized Steiner Tree, by the aubadditivity and monotonicity of the cost function in the Generalized Steiner Tree problem. The inverse is not true in general, and this reduction invokes a fee in the approximation ratio, given in the following definition.

Definition 6. *Given an instance I of* Generalized Steiner Tree *let $\tau_k(I)$ denote the minimum cost of a connected spanning sub-hypergraph of $\mathcal{H}_k$. The k-ratio for a class $\mathcal{I}$ of* Generalized Steiner Tree *instances is defined by $\alpha_k = \sup_{I \in \mathcal{I}} \frac{\tau_k(I)}{\mathsf{opt}(I)}$.*

Note that for $\mathcal{I}$ being the class of ST-MSP instances, α_2 is the parameter α defined in the introduction, and that by [18] we have $\alpha_2 = \alpha = \Delta - 1$. We have $\alpha_k = 1$ for instances with $|R| = k$, and in general α_k is monotone decreasing and approaching 1 when k becomes larger.

In Section 2.1 we prove the following statement, which is of independent interest, and may find applications in other network design problems.

Theorem 3. *There exists polynomial time algorithm that given a hypergraph $\mathcal{H} = (R, \mathcal{E})$ with hyper-edge cost $\{c(A) : A \in \mathcal{E}\}$ and a spanning tree T^* of (edges of size 2 of) $\mathcal{H}$ computes a spanning connected sub-hypergraph $\mathcal{T}$ of $\mathcal{H}$ of cost at most $\tau \left(1 + \ln \frac{c(T^*)}{\tau}\right)$, where τ is the minimum-cost of a connected spanning sub-hypergraph of $\mathcal{H}$.*

Corollary 1. *For any constant k,* Generalized Steiner Tree *admits an approximation ratio $\alpha_k (1 + \ln \alpha_2)$, provided that for any $A \in \mathcal{E}_k$, the instance with the terminal set A can be solved in polynomial time.*

Proof. By the assumptions, the hypergraph $\mathcal{H}_k$, and the costs $c^*(A)$ with the corresponding trees T_A for $A \in \mathcal{E}_k$, can be computed in polynomial time. We can also compute in polynomial time an optimal spanning tree T^* in $\mathcal{H}_2$; note that $c(T^*) \leq \alpha_2 \cdot \mathsf{opt}$. Then we apply the algorithm in Theorem 3 to compute a sub-hypergraph $\mathcal{T}$ of $\mathcal{H}_k$ of c^*-cost at most $\tau \left(1 + \ln \frac{c(T^*)}{\tau}\right)$, where τ is the minimum-cost of a connected spanning sub-hypergraph of $\mathcal{H}_k$. Let opt denote the optimal solution value for the Generalized Steiner Tree instance. Note that $\mathsf{opt} \leq \tau \leq \alpha_k \mathsf{opt}$. Let $T = \cup_{A \in \mathcal{T}} T_A$. Since $\mathcal{T}$ is a connected hypergraph, T is a feasible solution to the Generalized Steiner Tree instance. We have $c(T) \leq \sum_{A \in \mathcal{T}} c(T_A) = c^*(\mathcal{T})$, by the monotonicity and the subadditivity of the c-costs. Thus we have:

$$c(T) \leq c^*(\mathcal{T}) \leq \tau \left(1 + \ln \frac{c(T^*)}{\tau}\right) = \tau \left(1 + \ln \frac{c(T^*)/\mathsf{opt}}{\tau/\mathsf{opt}}\right) \leq \alpha_k \mathsf{opt} \left(1 + \ln \alpha_2\right) \ .$$

$\square$

Du and Zhang [7] showed that for the classic Steiner Tree problem, $\alpha_k \leq 1 + 1/\lfloor \lg k \rfloor$, where $\lg k = \log_2 k$ denotes logarithm base 2. In Section 4 we prove the following.

Theorem 4. *For* ST-MSP, $\alpha_k \leq 1 + \frac{2}{\lfloor \lg \lfloor k/(\Delta - 1) \rfloor \rfloor}$ *for any integer* $k \geq 2\Delta - 2$.

Note that $k \geq \Delta$ is necessary if we want $\alpha_k < 2$. Otherwise, for an instance I of Δ points on the unit ball we have $\frac{\tau(I)}{\mathsf{opt}(I)} = \frac{k}{\Delta}$, so $\alpha_k \geq \frac{k}{\Delta}$ if $k \leq \Delta$.

From Corollary 1 and Theorem 4 we conclude that for any constant $k \geq 2\Delta - 2$, it is possible to compute in polynomial time a solution to an ST-MSP instance of size at most $\alpha_k (1 + \ln(\Delta - 1)) \mathsf{opt}$, where α_k is as in Theorem 4. For the metric space $\mathbb{R}^2$, and given a constant $\epsilon > 0$ let $k = 2^{O(1/\epsilon)}$ with sufficient large constant. Then by Theorem 4, $\alpha_k \leq 1 + \epsilon / (1 + \ln 4)$, and the approximation ratio of our algorithm is $1 + \ln 4 + \epsilon$. This completes the proof of Theorem 1.

2.1 Proof of Theorem 3

For the proof of Theorem 3 we need the following definition.

Definition 7. *Given a tree* $T = (R, F)$ *we say that* $A \subseteq R$ *overlaps* $F' \subseteq F$ *if the graph obtained from* $T \setminus F'$ *by shrinking* A *into a single node is a tree. Given edge cost* $\{c(e) : e \in F\}$ *let* $F(A)$ *be a maximum cost edge set overlapped by* A.

Note that $F \setminus F(A)$ is an edge set of a minimum cost spanning tree in the graph obtained from T by shrinking A into a single node; hence $F(A)$ can be computed in polynomial time. The following statement appeared in [24] (see also [2]); we provide a proof for completeness of exposition.

Lemma 1. *Let* $T = (R, F)$ *be a tree with edge costs* $\{c(e) : e \in F\}$ *and let* $(R, \mathcal{E})$ *be a connected hypergraph. Then* $\sum_{A \in \mathcal{E}} c(F(A)) \geq c(F)$. *Thus there exists* $A \in \mathcal{E}$ *such that*

$$\frac{c(F(A))}{c(A)} \geq \frac{c(F)}{c(\mathcal{E})} \ .$$

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Proof. For a node $v \in A$, let C_v be the connected component in $T \setminus F(A)$ that contains v. For an edge $e \in F(A)$ that connects two components C_u, C_v, let $y(e) = uv$ be the replacement edge of e, of cost $c(y(e)) = c(e)$. The graph $T \cup \{y(e)\}$ contains a single cycle and $y(e)$ is the heaviest edge in this cycle, since otherwise $F(A)$ is not minimal. For a hyperedge $A \in \mathcal{E}$ let $y(A) = \cup_{e \in F(A)} y(e)$ be the replacement set of A, and let $y(\mathcal{E}) = \cup_{A \in \mathcal{E}} y(A)$. It is easy to see that $y(A)$ span A, and $y(\mathcal{E})$ span R. Consider a MST on $T \cup y(\mathcal{E})$. By the cycle property of a MST, no edge from $y(\mathcal{E})$ would participate in that MST, so $c(T) \leq c(y(\mathcal{E}))$. Finally, $c(y(\mathcal{E})) = \sum_{A \in \mathcal{E}} y(A) = \sum_{A \in \mathcal{E}} c(F(A))$, and the lemma follows. $\qquad\square$

Local Replacement Algorithm
Input: A hypergraph $\mathcal{H} = (R, \mathcal{E})$ with hyper-edge cost $\{c(A) : A \in \mathcal{E}\}$, and a spanning tree $T^* = (R, F^*)$ of (edges of size 2 of) $\mathcal{H}$.
Initialization: $\mathcal{J} \leftarrow \emptyset$, $F \leftarrow F^*$, $T \leftarrow (R, F)$.
While $c(F) > 0$ do:
       Find $A \in \mathcal{E}$ with $\frac{c(F(A))}{c(A)}$ maximum.
    - *If* $c(F(A)) > c(A)$ then do:
        - Update $T, \mathcal{H}$: remove $F(A)$ and shrink A into a single node.
        - $F \leftarrow F \setminus F(A)$ and $\mathcal{J} \leftarrow \mathcal{J} \cup \{A\}$.
    - *Else* STOP and *Return* $\mathcal{T} = (R, F \cup \mathcal{J})$.
EndWhile
Return $\mathcal{T} = (R, F \cup \mathcal{J})$.

At every iteration $|F|$ decreases by at least 1, hence the algorithm runs in polynomial time, and clearly it computes a feasible solution. We prove the approximation ratio. Let F_i and $\mathcal{J}_i$ be the set stored in F and $\mathcal{J}$, respectively, at the beginning of iteration $i+1$, and let A_i be the hyperedge picked at iteration i. Denote $f_i = c(F_i)$ and $s_i = c(A_i)$, and recall that τ denotes the minimum cost of a connected spanning sub-hypergraph of $\mathcal{H}$. At iteration i we remove $F_{i-1}(A_i)$ from F_{i-1} after verifying that $c(F_{i-1}(A_i)) > c(A_i) = s_i$. Hence

$$f_i \leq f_{i-1} - \max\{c(F_{i-1}(A_i)), c(A_i)\} = f_{i-1} - s_i \cdot \max\left\{\frac{c(F_{i-1}(A_i))}{c(A_i)}, 1\right\}$$

By Lemma 1, $\frac{c(F_{i-1}(A_i))}{c(A_i)} \geq \frac{f_{i-1}}{\tau}$. Thus we have

$$f_i \leq f_{i-1} - s_i \cdot \max\{f_{i-1}/\tau, 1\} \ . \tag{1}$$

The algorithm stops if either $c(F_q) = 0$ or $c(F(A)) \leq c(A)$ at iteration $q+1$. In the latter case, $1 \geq c(F_q)/\tau$ follows by Lemma 1. In both cases, we have that there exists an index q such that $f_{q-1} > \tau \geq f_q$ holds. Now we use the following statement from [6].

Lemma 2. *Let $\tau > 0$ and $f_0, \ldots, f_q$ and $s_1, \ldots, s_q$ be sequences of positive reals satisfying $f_0 > \tau \geq f_q$, such that (1) holds. Then $f_q + \sum_{i=1}^{q} s_i \leq \tau(1 + \ln(f_0/\tau))$.*

Let q be an index such that $f_{q-1} > \tau \geq f_q$ holds. We may assume that $f_0 = c(F^*) > \tau > 0$. Note that $c(\mathcal{J}_q) = \sum_{i=1}^{q} s_i$ and that $c(F_i) + c(\mathcal{J}_i) \leq c(F_{i-1}) + c(\mathcal{J}_{i-1})$ for any i. Hence from Lemma 2 we conclude that

$$c(\mathcal{T}) \leq c(F_q) + c(\mathcal{J}_q) = f_q + \sum_{i=1}^{q} s_i \leq \tau(1 + \ln(f_0/\tau)) = \tau\left(1 + \ln\frac{c(T^*)}{\tau}\right) \ .$$

This finishes the proof of Theorem 3.

3 Proof of Theorem 2

To illustrate our idea, we first prove Theorem 2 for a particular simple case – the Steiner Forest with Minimum Number of Steiner Points (SF-MSP) problem, when $r_{uv} \in \{0, 1\}$.

Definition 8. *For a subset C of nodes of a graph $G = (V, E)$ let us use the following notation: $\Gamma_G(C)$ is the set of neighbors of C in G; $\delta_G(C) = \delta_E(C)$ is the set of edges in E with exactly one endnode in C; $E(C)$ is the set of edges in E with both endnodes in C. Given $R \subseteq V$, an R-component of G is a subgraph of G with node set $C \cup \Gamma_G(C)$ and edge set $E(C) \cup \delta_G(C)$, where C is a connected component of $G \setminus R$.*

The cut-LP relaxation for Steiner Forest is:

$$\tau^* = \min \quad \sum_{e \in E} c_e x_e$$
$$\text{s.t.} \quad \sum_{e \in \delta_E(Y)} x_e \geq f(Y) \qquad \forall \, \emptyset \neq Y \subset V$$
$$0 \leq x_e \leq 1 \qquad \forall e \in E$$

where $f(Y) = 1$ if there are $u, v \in V$ with $r_{uv} = 1$ and $|\{u, v\} \cap Y| = 1$, and $f(Y) = 1$ otherwise.

Robins and Salowe [22] proved that if V is a set of ponts in a metric space, then there exists a tree $T = (V, E)$ of minimum total length $\sum_{uv \in E} d(u, v)$ that has maximum degree $\leq \Delta$. Since any inclusion-minimal solution to a Steiner Forest instance is a forest, this implies the following.

Lemma 3. *For any instance of SF-MSP there exists an optimal solution S, G such that G has maximum degree Δ.* $\square$

The following statement was first observed in [13].

Lemma 4. *Let R be a set of terminals and S a set of points in a normed space such that the unit-disc graph of $R \cup S$ contains a tree T with leaf set R. Let S' be obtained from S by replacing each $v \in S$ by $\deg_T(v)$ copies of v. Then the unit disc graph of $R \cup S'$ contains a simple cycle on $R \cup S'$.*

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Proof. Traverse the tree T in a DFS order; each time a node $v \in S$ is visited, choose a different copy of v. $\qquad\square$

Given a tree T, we will call a cycle as in the lemma above a *DFS cycle* of T.

Now we can prove Theorem 2 for the SF-MSP case. Let S be an inclusion minimal solution to an SF-MSP instance. By Lemma 3, the unit-disc graph of $R \cup S$ contains an r-connected forest H such that $\deg_H(v) \leq \Delta$ for every $v \in S$. Every R-component T of H (a.k.a. full Steiner component) is a tree with leaf set in R and all internal nodes in S. It is easy to see that by replacing every R-component T of H by a DFS cycle of capacity $1/2$ results in a feasible solution to the cut-LP relaxation, which proves Theorem 2 for the SF-MSP case.

Now we prove Theorem 2 for $\{0,1,2\}$-SN-MSP. We start by describing the cut-LP relaxation for SN. We need some definitions.

Definition 9. *An ordered pair $\hat{X} = (X, X^+)$ of subsets of a groundset V is called a* biset *if $X \subseteq X^+$; X is the* inner part *and X^+ is the* outer part *of $\hat{X}$, $\Gamma(\hat{X}) = X^+ \setminus X$ is the* boundary *of $\hat{X}$, and $X^* = V \setminus X^*$ is the* complementary *set of $\hat{X}$. An edge $e = uv$* covers *a biset $\hat{X}$ if it has one endnode in X and the other in $V \setminus X^+$. For a biset $\hat{X}$ and an edge-set/graph J let $\delta_J(\hat{X})$ denote the set of edges in J covering $\hat{X}$.*

By Menger's Theorem, a graph $G = (V, E)$ is (r, Q)-connected if, and only if, $|\delta_E(\hat{Y})| \geq f(\hat{Y})$, where f is a biset-function defined by

$$f(\hat{Y}) = \begin{cases} \max_{uv \in \delta_{D_r}(\hat{Y})} r_{uv} - |\Gamma(\hat{Y})| & \text{if } \Gamma(\hat{Y}) \subseteq Q \\ 0 & \text{otherwise} \end{cases}$$

The cut-LP relaxation for SN is

$$\begin{aligned} \tau^* = \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta_E(\hat{Y})} x_e \geq f(\hat{Y}) && \forall \text{ biset } \hat{Y} \\ & 0 \leq x_e \leq 1 && \forall e \in E \end{aligned}$$

We will say that a graph with edge capacities x_e is *fractionally (r, Q)-connected* if x is a feasible solution to the above cut-LP relaxation.

To prove Theorem 2, we prove in the next sections the following two theorems about $\{0,1,2\}$-connected graphs, that are of independent interest, and may find further applications in low connectivity network design. An r-connected graph G is *minimally r-connected* if no proper subgraph of G is r-connected.

Theorem 5. *Let G be a minimally (r, Q)-connected graph such that $Q \cup R = V$ and $r_{uv} \in \{0,1,2\}$ for all $u, v \in R$. Then every R-component is a tree. Furthermore, for any subset $\mathcal{C}$ of connected components of $G \setminus R$, replacing for each $C \in \mathcal{C}$ the corresponding tree by a DFS cycle of capacity $1/2$ results in a fractionally (r, Q)-connected graph.*

Theorem 6. *Let R be a set of terminals in a normed space, let $B \subseteq R$, and let r be a $\{0,1,2\}$ requirement function on R. Let S be an inclusion minimal set of points such that the unit-disc graph of $R \cup S$ is $(r, B \cup S)$-connected. Among all $(r, B \cup S)$-connected spanning subgraphs of the unit-disc graph of $R \cup S$, let $G = (V, E)$ be one of minimum total length $\sum_{uv \in E} d(u,v)$. Then $\deg_G(v) \leq \Delta$ for all $v \in S$.*

Particular cases of Theorem 6 were proved by Robins and Salowe [22] for $r \equiv 1$, and by Calinescu [3] for $r \equiv 2$. We prove Theorems 5 and 6 in Sections 3.1 and 5, respectively, relying on these particular cases. From Theorem 5, Theorem 6, and Lemma 4, we obtain the following corollary, that implies Theorem 2.

Corollary 2. *For any feasible solution S, G to an instance of $\{0,1,2\}$-SN-MSP there exists a half integral bead solution of value at most $\Delta|S|/2$.*

3.1   Proof of Theorem 5

A *block* of a graph G is an inclusion-maximal 2-connected subgraph of G, or a graph induced by a bridge of G. It is known that every edge belongs to exactly one block, hence the blocks of a graph partition its edge set. Furthermore, any two blocks have at most one node in common.

Lemma 5. *Let $G = (V, E)$ be a minimally (r, Q)-connected graph such that $r_{uv} \in \{0,1,2\}$ for all $uv \in D_r$ and $Q \cup R = V$. Let $G' = (V', E')$ be a 2-connected block of G and let $R' = R \cap V'$. Then $|R \cap V'| \geq 2$ and no proper 2-connected subgraph of G' that contains R' exists.*

Proof. We may assume that G is connected, as otherwise we may consider each connected component of G separately. Any V'-component C has exactly one node in V', which we call the *attachment node* of C. Note that if $r_{uv} = 2$ such that v belongs to a V'-components C_v of G and $u \notin C_v$, then the attachment node of C_v is in $V \setminus Q$, and hence is in R, by the assumption $Q \cup R = V$.

We prove that $|R'| \geq 2$ Since G' is 2-connected, and G is minimally (r, Q)-connected, there exists $uv \in D_r$ with $r_{uv} = 2$ such that $u \in V'$, or u, v belong to disjoint V'-components. Suppose that $u \in V'$. If $v \in V'$ then we are done. Else, v belongs to a V'-component, and the attachment node of this component is in R. If u, v belong to disjoint V'-components, then the attachment nodes of these components are distinct and belong to R. In all cases, we have $|R'| \geq 2$.

We prove that if $G'' = (V'', E'')$ is a 2-connected subgraph of G' that contains R', then $G'' = G'$. Suppose that $G'' \neq G'$. Let A be the set of attachment nodes that are in $V' \setminus V''$. Note that $A \subseteq Q \setminus R$. In G', shrink V'' into a single node v'', and take F to be the edge set of some inclusion minimal tree in G' that contains $A \cup \{v''\}$. Let $I = E'' \cup F$. If $A = \emptyset$ then $F = \emptyset$, and $I = E''$. Otherwise, there is $a \in A$ that has degree exactly 1 in (V', I). In both cases, I must be a proper subset of $E' \setminus E''$. Let $\hat{G}$ be obtained from G by replacing E' by I. It is not hard to verify that $\hat{G}$ is (r, Q)-connected, since $A \subseteq Q \setminus R$. Furthermore, $\hat{G}$ is a proper subgraph of G, since I is a proper subset of $E' \setminus E''$ This contradicts the minimality of G. $\qquad\square$

A path P is an *L-chord path* of a cycle L in a graph G if the endnodes of P are in L but no internal node of P is in L. Relying on ear decomposition of 2-connected graphs, Calinescu [3] proved the following.

Lemma 6 ([3]). *Let $G' = (V', E')$ be a 2-connected graph and let $R' \subseteq V$ with $|R'| \geq 2$. Suppose that no proper 2-connected subgraph of G that contains R' exists. Then any cycle L in G' contains at least 2 nodes in R', and any L-chord path contains at least one node in R' that does not belong to L.*[1]

We generalize this to $\{0, 1, 2\}$-Q-connectivity, as follows.

Lemma 7. *Let $G = (V, E)$ be a minimally (r, Q)-connected graph such that $r_{uv} \in \{0, 1, 2\}$ for all $uv \in D_r$ and $Q \cup R = V$. Then any cycle L in G contains at least 2 nodes in R, and any L-chord path contains at least one node in R that does not belong to L.*

Proof. Let L be a cycle in G. Then L is contained in some 2-connected block $G' = (V', E')$ of G; moreover, any L-chord path is also contained in G'. Let $R' = R \cap V'$. By Lemma 5, G', R' satisfy the conditions of Lemma 6; hence the statement follows from Lemma 6. □

By Lemma 7, the graph $G \setminus R$ is a forest, and every $v \in R$ has at most one neighbor in each connected component of $G \setminus R$. This implies the first part of Theorem 5. Now we prove the second part, namely, the following.

Lemma 8. *Let G be a minimally (r, Q)-connected graph such that $r_{uv} \in \{0, 1, 2\}$ for all $uv \in D_r$ and $Q \cup R = V$. Then for any subset $\mathcal{C}$ of connected components of $G \setminus C$, replacing for each $C \in \mathcal{C}$ the corresponding tree T_C by a DFS cycle on $\Gamma_G(C)$ of capacity $1/2$ results in a fractionally (r, Q)-connected graph H.*

Proof. Suppose to the contrary that there exists $uv \in D_r$ such that u, v are not fractionally (r_{uv}, Q)-connected in H. This may happen only if $r_{uv} = 2$ and there exists $C \in \mathcal{C}$ such that u, v can be disconnected by removing two elements a, b of T_C from G. If one of a, b is an edge we can replace it by its endnode in T_C, hence we may assume that each of a, b is a node. Note that $a \neq b$, since otherwise u, v can be disconnected by removing the single element a, contradicting that $\lambda_G^Q(u, v) \geq r_{uv} = 2$. Let P_{ab} be the ab-path in T_C. Note that all the internal nodes of P_{ab} are in C, so none of them is a terminal. Consider two $(Q \cup E)$-disjoint u, v paths in G. One of them must contain a and the other contains b; denote these paths by P_a and P_b, respectively. The union of the paths P_a and P_b contains a simple cycle L that contains a, b. Hence the path P_{ab} has a subpath P such that P is an L-chord path. This contradicts Lemma 7, since no internal node of P is a terminal. □

The proof of Theorem 5 is complete.

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[1] This statement is not true for edge-connectivity; for example, if $R = \{s, t\}$ and G consists of 2 edge-disjoint st-paths that have 2 nodes u, v in common, then the simple cycle that contains u, v contains no node from R.

4 Proof of Theorem 4

For a tree $T = (V, F)$ and $A \subseteq V$ let $T_A = (V_A, F_A)$ be the inclusion minimal subtree of T that contains A. To prove Theorem 4 it is sufficient to prove the following.

Lemma 9. *Let $T = (V, F)$ be a tree of maximum degree $\Delta \geq 2$, let $R \subseteq V$, and let $S = V \setminus R$. Then for any integer $k \geq 2\Delta - 2$ there exists a connected hypergraph $\mathcal{H} = (R, \mathcal{E})$ of rank $\leq k$ such that $\sum_{A \in \mathcal{E}} |V_A \cap S| \leq \left(1 + \frac{2}{\lfloor \lg \lfloor k/(\Delta-1)\rfloor \rfloor}\right)|S|.$*

To prove Lemma 9 we prove the following.

Lemma 10. *Let $T = (V, F)$ be a tree with edge costs $\{c(e) \geq 1 : e \in F\}$ and let $R \subseteq V$. Then for any integer $p \geq 2$ there exists a connected hypergraph $\mathcal{H} = (R, \mathcal{E})$ of rank $\leq p$ such that $\sum_{A \in \mathcal{E}} c(F_A) + |\mathcal{E}| - 1 \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right)c(T).$*

Lemma 10 will be proved later. Now we show that it implies Lemma 9. An *R-component* of T is a maximal inclusion subtree of T such that all its leaves are in R but no its internal node is in R. It is easy to see that it is sufficient to prove Lemma 9 for each R-component separately, hence we may assume that R is the set of leaves of T.

If T is a star, then since $k \geq 2\Delta - 2 \geq \Delta$, we let $\mathcal{E}$ to consist of a single hyperedge $A = R$. Then $|V_A \cap S| = 1 = |S|$, and Lemma 9 holds in this case.

Henceforth assume that T is not a star. For $v \in S$ let $R(v)$ be the set of neighbors of v in R, and note that $|R(v)| \leq \Delta - 1$. Let $T' = (V', F') = T \setminus R$ and let $R' = \{v \in S : R(v) \neq \emptyset\}$. Applying Lemma 10 on T' with unit edge-costs and R', we obtain that for $p = \lfloor k/(\Delta - 1)\rfloor$ there exists a connected hypergraph $\mathcal{H}' = (R', \mathcal{E}')$ of rank $\leq p$ such that $\sum_{A' \in \mathcal{E}'} |F'_{A'}| + |\mathcal{E}'| - 1 \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right)|F'|.$ Note that $|F'| = |V'| - 1$ and that $|V'_{A'}| = |F'_{A'}| - 1$ for every $A' \in \mathcal{E}'$. Hence

$$\sum_{A' \in \mathcal{E}'} |V'_{A'}| - 1 \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right)(|V'| - 1) \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right)|V'| - 1.$$

For $A' \in \mathcal{E}'$ let $A = \cup_{v \in A'} R(v)$; then $|A| \leq p(\Delta - 1)$. Let $\mathcal{E} = \{A : A' \in \mathcal{E}'\}$. Then $\mathcal{H} = (R, \mathcal{E})$ is a connected hypergraph of rank $\leq p(\Delta - 1) \leq k$, and

$$\sum_{A \in \mathcal{E}} |V_A \cap S| = \sum_{A' \in \mathcal{E}'} |V'_{A'}| \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right)|V'| = \left(1 + \frac{2}{\lfloor \lg \lfloor k/(\Delta - 1)\rfloor \rfloor}\right)|S|.$$

In the rest of this section we prove Lemma 10, by extending the proof of Du and Zhang [7] of an existence of a connected hypergraph $\mathcal{H} = (R, \mathcal{E})$ of rank $\leq p$ such that $\sum_{A \in \mathcal{E}} c(F_A) \leq \left(1 + \frac{1}{\lfloor \lg p \rfloor}\right)c(T)$. We have an extra term of $|\mathcal{E}| - 1$, and we show that this term can be bounded by $\frac{c(T)}{\lfloor \lg p \rfloor}$.

We start by transforming the tree into a (rooted) binary tree T with edge-costs, which node set is partitioned into a set R of terminals and a set S of non-terminals, such that the following properties hold:

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(A) R is the set of leaves of T.

(B) The cost of any edge of T is either 0 or is at least 1, and among the edges that connect a node in $S = V \setminus R$ to its children, at most one has cost 0.

(C) T is a full binary tree, namely, every $v \in S$ has exactly 2 children.

To obtain such a tree, root T at an arbitrary non-leaf node $\hat{s} \in S = V \setminus R$, and apply the following standard reductions.

1. While T has a leaf in S, remove this leaf; hence every leaf of T is in R. Then, for every $v \in R$ that is not a leaf, add to T a new node v' and an edge vv' of cost 0, add v' to R, and move v from R to S. After this step, properties (A) and (B) hold.

2. While there is $v \in S$ that has one child, replace the path P of length 2 that contains v by a single edge of cost $c(P)$, and exclude v from S. After this step, every $v \in S$ has at least 2 children.

3. While there is $v \in S$ that has more than 2 children, do the following. Let u be a child of v such that the cost of the edge vu is at least 1. Add a new node v' and the edge vv' of cost 0, and for every child of u' of v distinct from u replace the edge vu' by the edge vu'. After this step, all the three properties (A), (B), and (C) hold.

Consequently, to prove Lemma 10, it is sufficient to prove the following.

Lemma 11. *Let $T = (V, F)$ be a tree with edge costs $\{c(e) : e \in F\}$ and leaf set R, satisfying properties (A),(B),(C), Then for any integer $p \geq 2$ there exists a connected hypergraph $\mathcal{H} = (R, \mathcal{E})$ of rank $\leq p$ such that $\sum_{A \in \mathcal{E}} c(F_A) + |\mathcal{E}| - 1 \leq \left(1 + \frac{2}{\lceil \lg p \rceil}\right) c(T)$.*

Let $T = (V, F)$ be a rooted tree with leaf set R and let $S = V \setminus R$. For two nodes u, v of T let $P_T(u, v)$ denote the unique path in T between u and v.

Definition 10. *We say that T is* proper *if every node in S has at least 2 children. We say that a mapping $f : S \to R$ is T-proper if*

– *For every $u \in S$, $f(u)$ is a descendant of u.*

– *The paths $\{P_T(u, f(u)) : u \in S\}$ are edge disjoint.*

Given a subtree T' of T with leaf set L' and a proper mapping f, the set of terminal connecting paths of T' is $\{P_T(u, f(u)) : u \in L' \setminus R\}$. Let $\hat{T}'$ denote the tree obtained from T' by adding to T' all the terminal connecting paths.

Du and Zhang [7] proved that any proper tree T admits a proper mapping. We prove the following.

Lemma 12. *Let $T = (V, F)$ be a proper tree and let $F_1 \subseteq F$ be such that any $u \in S$ has a child connected to u by an edge in F_1. Then there exists a T-proper mapping f such that for every $u \in S$, the path $P_T(u, f(u))$ contains at least one edge in F_1.*

Proof. The proof is by induction on the height of the tree. Let T be a tree as in the lemma of height h. If $h = 1$, then T has one internal node (the root), say u, and we set $f(u)$ to be the node that is connected to u by an edge in F_1. Suppose that the statement is true for trees with height $h - 1 \geq 1$, and we prove it for trees of height h. Let T' be obtained from T by removing nodes of distance h from the root. By the induction hypothesis, for T' there exists a mapping f' as in the lemma. Let u be an internal node of T. Consider two cases.

Suppose that u is an internal node of T'. If $f'(u)$ is a leaf of T, then define $f(u) = f'(u)$. If $f'(u)$ is an internal of T, then $f'(u)$ is a leaf of T', and all its children in T are leaves. Then we set $f(u)$ to be a child of $f'(u)$ that is connected to $f'(u)$ by an edge in F_1

Suppose that u is a leaf of T'. Then the children of u in T are leaves, and we set $f(u)$ to be a child of u that is connected to u by an edge in F_1.

It is easy to verify that the obtained mapping f meets the requirements. □

The following statement is implicitly proved by Du and Zhang [7].

Lemma 13 ([7]). *Let T be a proper binary tree with non-negative edge costs and let f be a proper mapping. Then for any integer $p \geq 2$ there exists an edge-disjoint partition $\mathcal{T}$ of T into subtrees such that the following holds:*

(i) *The hypergraph with node set R and hyperedge set $\mathcal{E} = \{\hat{T}' \cap R : T' \in \mathcal{T}\}$ is connected and has rank at most p.*

(ii) *The total number of terminal connecting paths of all subtrees in $\mathcal{T}$ is at least $|\mathcal{T}| - 1$, and their total cost is at most $c(T)/\lfloor \lg p \rfloor$.*

We now finish the proof of Lemma 11, and thus also of Lemma 10. Let $F_1 = \{e \in F : c(e) \geq 1\}$ and let f be a proper mapping as in Lemma 12. Let $\mathcal{T}$ be a partition as in Lemma 13, and let $\mathcal{E}$ be as in Lemma 13(i), so the hypergraph $\mathcal{H} = (R, \mathcal{E})$ is connected and has rank at most p. By Lemma 13(ii), the total number of terminal connecting paths of all subtrees is at least $|\mathcal{T}| - 1 = |\mathcal{E}| - 1$, while their total cost is at most $c(T)/\lfloor \lg p \rfloor$. Every terminal connecting path contains an edge from F_1, by Lemma 12, and thus has cost at least 1. Hence the total cost of all terminal connecting paths is at least $|\mathcal{E}| - 1$. Consequently

$$|\mathcal{E}| - 1 \leq \frac{c(T)}{\lfloor \lg p \rfloor} .$$

For $A = \hat{T}' \cap R \in \mathcal{E}$ let $P(T')$ denote the union of the edge sets of the terminal connecting paths of T'. Then $c(F_A) \leq c(\hat{T}') = c(T) + c(P(T'))$, hence

$$\sum_{A \in \mathcal{E}} c(F_A) \leq \sum_{T' \in \mathcal{T}} [c(T') + c(P(T'))] = \sum_{T' \in \mathcal{T}} c(T') + \sum_{T' \in \mathcal{T}} c(P(T')) \leq c(T) + \frac{c(T)}{\lfloor \lg p \rfloor} .$$

Summarizing, we have

$$\sum_{A \in \mathcal{E}} c(F_A) + |\mathcal{E}| - 1 \leq c(T) + \frac{c(T)}{\lfloor \lg p \rfloor} + \frac{c(T)}{\lfloor \lg p \rfloor} = \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right) c(T) .$$

The proof of Lemma 11, and thus also of Lemma 10 and Theorem 4 is now complete.

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5 Proof of Theorem 6

To prove Theorem 6, we use the following result of Calinescu [3].

Lemma 14 ([3]). *Let R' be a set of terminals in a normed space and let S' be an inclusion minimal set of points such that the unit-disc graph of $R' \cup S'$ is 2-connected. Among all 2-connected spanning subgraphs of the unit-disc graph of $R' \cup S'$, let $G' = (V', E')$ be one of minimum total length $\sum_{uv \in E'} d(u, v)$. Then $\deg_{G'}(v) \leq \Delta$ for all $v \in S'$.*

Let G, B, S, r be as in Theorem 6. As in the proof of Theorem 5, we may assume that G is connected. Consider a 2-connected block $G' = (V, E')$ of G. Let $R' = R \cap V'$ and $S' = S \cap V'$. Then by Lemma 5 no proper 2-connected subgraph of G' that contains R' exists, hence S' is an inclusion minimal set of points such that the unit-disc graph of $R \cup S$ is 2-connected. Furthermore, since G has minimum total length, so is G'. Thus by Lemma 14, $\deg_{G'}(v) \leq \Delta$ for all $v \in S'$. Consequently, $\deg_G(v) \leq \Delta$ holds for any $s \in S$ that belongs to exactly one block of G. A node s is a *cut-node* of a connected graph if its removal disconnects the graph. It is known that s is a cut-node of a graph if and only if s belongs to at least two blocks of the graph. Our goal now is to show that $\deg_G(v) \leq \Delta$ holds for any cut-node $s \in S$ of G.

Let $s \in S$ be a cut-node of G. Suppose to the contrary that $\deg_G(v) \geq \Delta + 1$. Then by [22] there are neighbors a, b of s in G such that $d(a, b) \leq d(a, s)$. By a reduction from [22,3], we may assume that all the lengths of the edges in G are distinct, hence $d(a, b) < d(a, s)$. Let H be obtained from G by replacing the edge sa by the edge ab. We claim that H is (r, Q)-connected, which gives a contradiction, since H has smaller total length than G. Thus to finish the proof of Theorem 6, it is sufficient to prove the following.

Lemma 15. *Let $G = (V, E)$ be an (r, Q)-connected graph with $r_{uv} \in \{0, 1, 2\}$ for all $uv \in D_r$, and let $sa, sb \in E$ be a pair of (r, Q)-connectivity critical edges with $s \in Q \setminus R$. Then the graph H obtained from G by replacing the edge sa by the edge ab is also (r, Q)-connected.*

Proof. Suppose to the contrary that there is $xx' \in D_r$ such that $\lambda_H^Q(xx') \leq r_{xx'} - 1$. It is easy to see that any u, v that are connected in G also connected in H, hence we must have $r_{xx'} = 2$. Consider the graph $J = G \setminus \{sa\}$. Since $\lambda_{J \cup \{sb\}}^Q(x, x') = 1$ and $\lambda_{J \cup \{sa\}}^Q(x, x') = 2$, then by Menger's Theorem, there exists a biset $\hat{X}$ such that $s \in X$, $a \in X^*$, $b \in \Gamma(\hat{X}) \subseteq Q$, $\delta_G(\hat{X}) = \{sa\}$, and one of x, x' belongs to X and the other to X^*, say $x \in X$ and $x' \in X^*$; see Figure 5(a). Similarly, since the edge sb is (r, Q)-connectivity critical, there exist $yy' \in D_r$ with $r_{yy'} = 2$ and a biset $\hat{Y}$, such that $s \in Y$, $b \in Y^*$, $sb \in \delta_G(\hat{Y})$, $|\delta_G(\hat{Y})| + |\Gamma(\hat{Y})| = 2$, $\Gamma(\hat{Y}) \subseteq Q$, and one of y, y' belongs to Y and the other to Y^*, say $y \in Y$ and $y' \in Y^*$; see Figure 5(b). Now we consider the three cases, $a \in \Gamma(\hat{Y})$, $a \in Y^*$, and $a \in Y$, and at each of them arrive to a contradiction.
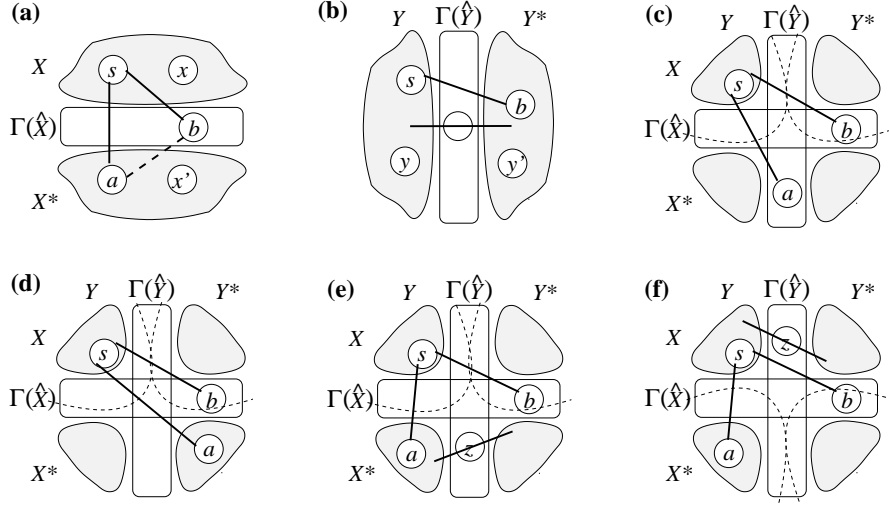
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Fig. 1. Illustration to the proof of Lemma 15.

Suppose that $a \in \Gamma(\hat{Y})$; see Figure 5(c). Then $x \notin \Gamma(\hat{Y})$, so $x \in X \cap Y$ or $x \in X \cap Y^*$. If $x \in X \cap Y^*$ then the biset $\hat{Z} = \hat{X} \setminus \hat{Y}$ satisfies $|\Gamma(\hat{Z})| + |\delta_G(\hat{Z})| = 1$ (since $\Gamma_G(\hat{Z}) = \{b\}$ and $\delta_G(\hat{Z}) = \emptyset$), $x \in Z$, and $x' \in Z^*$; this contradicts the assumption $\lambda_G^Q(x, x') = 2$. In the case $x \in X \cap Y$, we obtain a similar contradiction for the biset $\hat{Z} = (X \cap Y \setminus \{s\}, X \cap Y)$.

The analysis of the case $a \in Y^*$, see Figure 5(d), is similar to that of the case $a \in \Gamma(\hat{Y})$.

Now suppose that $a \in Y$; see Figure 5(e,f). Since $|\Gamma(\hat{Y})| + |\delta_G(\hat{Y})| = 2$ and since $sb \in \delta_G(\hat{Y})$, there is another element $z \in \Gamma(\hat{Y}) \cup \delta_G(\hat{Y})$. Note that if z is a node then $z \in X^* \cap \Gamma(\hat{Y})$ (Figure 5(e)) or $z \in X \cap \Gamma(\hat{Y})$ (Figure 5(f)). If z is an edge then z connects $Y \cap X^*$ and $Y^* \setminus X$ (Figure 5(e)) or $X \cap Y$ and $Y^* \setminus X^*$ (Figure 5(f)). In the cases in Figure 5(e), when $z \in X^* \cap \Gamma(\hat{Y})$ is a node, or z is an edge that connects $Y \cap X^*$ and $Y^* \setminus X$, the contradiction is obtained in the same way as in the case $a \in \Gamma(\hat{Y})$. We therefore are left with the cases in Figure 5(f), when $z \in X \cap \Gamma(\hat{Y})$ or z is an edge that connects $X \cap Y$ and $Y^* \setminus X^*$. Then we consider the location of x'. Note that $x' \notin \Gamma(\hat{Y})$, hence $x' \in Y$ or $x' \in Y^*$. In the case $x' \in Y$ we obtain a contradiction by considering the biset $\hat{Z} = \hat{Y} \setminus \hat{X}$, and in the case $x' \in Y^*$ we obtain a contradiction by considering the biset $\hat{Z} = \hat{X} \cup \hat{Y}$. $\qquad\square$

The proof of Theorem 6 is complete.

6 Conclusions

In this paper we considered the Survivable Network with Minimum Number of Steiner Points problem in a normed space. The main results of this paper are a

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$(1 + \ln(\Delta - 1) + \epsilon)$-approximation scheme for ST-MSP, and a Δ-approximation algorithm for $\{0, 1, 2\}$-SN-MSP. For ST-MSP in $\mathbb{R}^2$ this improves the ratio $2.5 + \epsilon$ of [5]. For $\{0, 1, 2\}$-SN-MSP, no nontrivial approximation algorithm was known before, but for the specific case of SF-MSP this improves the ratio 2Δ that can be deduced from the work of [13]. Obtaining even better approximation ratios is an important future work.

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